

# On large intersection and self-intersection local times in dimension five or more.

Amine Asselah

Université Paris-Est

Laboratoire d'Analyse et de Mathématiques Appliquées

UMR CNRS 8050

amine.asselah@univ-paris12.fr

## Abstract

We show a remarkable similarity between strategies to realize a large intersection or self-intersection local times in dimension five or more. This leads to the same rate functional for large deviation principles for the two objects obtained respectively by Chen and Mörters in [5], and by the present author in [1]. Also, we present a new estimate for the distribution of *high* level sets for a random walk, with application to the geometry of the intersection set of two *high* level sets of the local times of two independent random walks.

*Keywords and phrases:* self-intersection local times, large deviations, random walk.

*AMS 2000 subject classification numbers:* 60K35, 82C22, 60J25.

*Running head:* Intersection local times.

## 1 Introduction

Our purpose, in this paper, is twofold: (i) to unravel some connections between the rate functionals for the large deviations for intersection and self-intersection local times in dimension five or more, and (ii) to explore the geometry of intersection set of two level sets of the local times of two independent random walks.

To describe the two quantities we are comparing, a few notations are needed. Thus, we consider two independent aperiodic simple random walks on the cubic lattice  $\mathbb{Z}^d$ , with  $d \geq 5$ . More precisely, if  $S_n$  is the position of the first walk at time  $n \in \mathbb{N}$ , then  $S_{n+1}$  chooses uniformly at random a site of  $\{z \in \mathbb{Z}^d : |z - S_n| \leq 1\}$ , where for  $z = (z_1, \dots, z_d) \in \mathbb{Z}^d$ , the  $l^1$ -norm is  $|z| := |z_1| + \dots + |z_d|$ . When  $S_0 = x$ , we denote the law of this walk by  $P_x$ , and its expectation by  $E_x$ . All quantities related to the second walk differ with a tilde, whereas  $\mathbb{P} = P_0 \otimes \tilde{P}_0$ . For  $n \in \mathbb{N} \cup \{\infty\}$ , we denote the local times in a time period  $[0, n]$  by

$\{l_n(z), z \in \mathbb{Z}^d\}$  with  $l_n(z) = \sum_{k=0}^n \mathbb{1}\{S_k = z\}$ . The intersection local times of two random walks, in an infinite time horizon, is

$$\langle l_\infty, \tilde{l}_\infty \rangle = \sum_{z \in \mathbb{Z}^d} l_\infty(z) \tilde{l}_\infty(z), \quad \text{and} \quad \mathbb{E} \left[ \langle l_\infty, \tilde{l}_\infty \rangle \right] = \sum_{z \in \mathbb{Z}^d} G_d(z)^2 < \infty,$$

where the Green's function  $G_d$  is square summable in dimension 5 or more. On the other side, the self-intersection local times, in a time period  $[0, n]$ , is

$$\langle l_n, l_n \rangle = \sum_{z \in \mathbb{Z}^d} l_n^2(z), \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{E_0[\langle l_n, l_n \rangle]}{n} = 2G_d(0) - 1.$$

On one hand, Chen and Mörters in [5] have obtained a large deviations principle for  $\{\langle l_\infty, \tilde{l}_\infty \rangle \geq t\}$  for  $t$  large, in dimension 5 or more, by an elegant asymptotic estimation of the moments, improving on the pioneering work of Khanin, Mazel, Shlosman and Sinai in [7]. Their method provides a variational formula for the rate functional.

On the other hand, we have obtained a large deviation principle for  $\{\langle l_n, l_n \rangle - \mathbb{E}[\langle l_n, l_n \rangle] \geq \xi n\}$  for  $\xi > 0$  by a direct study of the contribution of each level set of the local times. Our approach provides information on the optimal strategy, but no information on the rate functional since it eventually relies on a subadditive argument.

Our first result establishes that in spite of a different time-horizon,  $\langle l_\infty, \tilde{l}_\infty \rangle$  and  $\langle l_n, l_n \rangle$  yield the same rate functions (up to a natural factor of 2).

**Proposition 1.1** *Consider a random walk in dimension 5 or more. Then, for any  $\xi > 0$*

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \log P(\langle l_n, l_n \rangle - E_0[\langle l_n, l_n \rangle] > \xi n) = -\mathcal{I}_{CM} \sqrt{\xi}, \quad (1.1)$$

where  $\mathcal{I}_{CM}$  is the rate functional for  $\langle l_\infty, \tilde{l}_\infty \rangle$ , obtained in variational form by Chen and Mörters in [5]. Thus, it is given in terms of an operator  $U_h$  by

$$\mathcal{I}_{CM} = \inf \{ \|h\|_2 : h \geq 0, \text{ and } \|U_h\| \geq 1 \},$$

where

$$U_h(f) = \sqrt{e^h - 1} (G_d - \delta_0) * (f \sqrt{e^h - 1}), \quad (1.2)$$

and  $\delta_0$  is the delta function at 0, and for  $z \in \mathbb{Z}^d$ ,  $f * g(z) = \sum_{y \in \mathbb{Z}^d} f(y)g(z - y)$ .

**Remark 1.2** We remark that Chen and Mörters' proof produces (and relies on) a finite volume version of their large deviation principle. Namely, for any finite subset  $\Lambda \subset \mathbb{Z}^d$ ,

$$\lim_{t \rightarrow \infty} \frac{1}{\sqrt{t}} \log \mathbb{P} \left( \sum_{z \in \Lambda} l_\infty(z) \tilde{l}_\infty(z) \geq t \right) = -2\mathcal{I}_{CM}(\Lambda), \quad \text{and} \quad \lim_{\Lambda \nearrow \mathbb{Z}^d} \mathcal{I}_{CM}(\Lambda) = \mathcal{I}_{CM}. \quad (1.3)$$

However, (1.3) gives no information on understanding which level sets of the local times is responsible for the large deviation.

Proposition 1.1 relies on the fact that both the excess self-intersection local times and large intersection local times are essentially realized on a finite region. The physical phenomenon behind this observation is that the sites which contribute in making  $\{\langle l_\infty, \tilde{l}_\infty \rangle > t\}$  are those for which both  $l_\infty(z)$  and  $\tilde{l}_\infty(z)$  are larger than  $\sqrt{t}/A$  for some constant  $A$ . In other words, define for  $\xi > 0$

$$\mathcal{D}(\xi) := \{z \in \mathbb{Z}^d : l_\infty(z) \geq \xi\}, \quad \text{and} \quad \tilde{\mathcal{D}}(\xi) := \{z \in \mathbb{Z}^d : \tilde{l}_\infty(z) \geq \xi\}.$$

Then, our next result reads as follows.

**Proposition 1.3** *Assume the lattice has five or more dimension. Then*

$$\limsup_{A \rightarrow \infty} \limsup_{t \rightarrow \infty} \frac{1}{\sqrt{t}} \log \mathbb{P} \left( \sum_{z \notin \mathcal{D}(\frac{\sqrt{t}}{A}) \cap \tilde{\mathcal{D}}(\frac{\sqrt{t}}{A})} l_\infty(z) \tilde{l}_\infty(z) > t \right) = -\infty. \quad (1.4)$$

Proposition 1.3 is based on the idea that  $\langle l_\infty, \tilde{l}_\infty \rangle$  is not *critical* in the sense that even when *weighting less* intersection local times, the strategy remains the same. In other words, define for  $q \leq 2$

$$\zeta(q) = \sum_{z \in \mathbb{Z}^d} l_\infty(z) \tilde{l}_\infty^{q-1}(z). \quad (1.5)$$

Then, we have the following Lemma, interesting on its own.

**Lemma 1.4** *Assume that  $d \geq 5$ . For any  $2 \geq q > \frac{d}{d-2}$ , there is  $\kappa_q > 0$  such that*

$$\mathbb{P}(\zeta(q) > t) \leq \exp(-\kappa_q t^{\frac{1}{q}}). \quad (1.6)$$

Our second task is to have some information on the geometry of the intersection of two level sets of the local times of two independent walks. For instance, we would like to address the following question: knowing the volume of the intersection of say  $\{z : l_\infty(z) \sim n\}$  and  $\{z : \tilde{l}_\infty(z) \sim n\}$  what can we say about the capacity of the intersection set?

To state our results, we need some more notations. For a finite subset  $\Lambda \subset \mathbb{Z}^d$ , we denote by  $\mathcal{S}(\Lambda)$  the set of permutations of the sites of  $\Lambda$ . Thus,  $\gamma \in \mathcal{S}(\Lambda)$  is written as  $\gamma = (\gamma(1), \dots, \gamma(|\Lambda|))$ , and we set  $\gamma(0) = 0$ . We define

$$T_\Lambda = \inf \{n > 0 : S_n \in \Lambda\}, \quad \text{and for } z \in \mathbb{Z}^d, \quad H(z) = \inf \{n \geq 0 : S_n = z\}.$$

Our key proposition is the following.

**Proposition 1.5** *Assume  $d \geq 3$ . There is a positive constant  $c_d$  such that for any  $\Lambda$  finite subset of  $\mathbb{Z}^d$ , and  $\{n(z), z \in \Lambda\}$  positive integers, we have*

$$P_0(l_\infty(z) = n(z), \forall z \in \Lambda) \leq (c_d \bar{n})^{|\Lambda|} (|\Lambda|!)^d e^{-\underline{n} \text{cap}(\Lambda)} \sum_{\gamma \in \mathcal{S}(\Lambda \setminus \{0\})} \prod_{i=1}^{|\Lambda \setminus \{0\}|} P_{\gamma(i-1)}(H(\gamma(i)) < \infty), \quad (1.7)$$

where we set

$$\bar{n} = \max_{z \in \Lambda} n(z), \quad \underline{n} = \min_{z \in \Lambda} n(z), \quad \text{and} \quad \text{cap}(\Lambda) = \sum_{z \in \Lambda} P_z(T_\Lambda = \infty).$$

**Remark 1.6** Proposition 1.5 is useful when dealing with *large level sets* of the local times. In other words, we need  $\underline{n} \times \text{cap}(\Lambda) \gg |\Lambda|$ . Since, we will see that  $\text{cap}(\Lambda) \geq c_d |\Lambda|^{1-2/d}$ , the range of applicability of (1.7) is  $\{(\underline{n}, |\Lambda|) : \underline{n} \gg |\Lambda|^{2/d}\}$ . We note that an alternative proof of Proposition 1.3 can be obtained using Proposition 1.5, in dimension 6 or more.

Let us now define more notations. For an integer  $n$ , let

$$\mathcal{L}(n) = \{z \in \mathbb{Z}^d : l_\infty(z) = n\}, \quad \text{and} \quad \tilde{\mathcal{L}}(n) = \{z \in \mathbb{Z}^d : \tilde{l}_\infty(z) = n\}. \quad (1.8)$$

With the same hypotheses as in Proposition 1.5, we have the following useful Corollary.

**Corollary 1.7** *Assume  $d \geq 5$ . There are positive constants  $\kappa_d, C_d$  such that for any positive integers  $n, m, L$ , we have*

$$P\left(|\mathcal{L}(n) \cap \tilde{\mathcal{L}}(m)| \geq L\right) \leq (C_d n \times m)^L (L!)^{2d} \exp\left(-\kappa_d (n+m) L^{1-\frac{2}{d}}\right). \quad (1.9)$$

**Remark 1.8** The inequality (1.9) is useful when  $n+m \gg L^{2/d}$ . A result in the direction of  $m$  and  $n$  small, is Lemma 2 of [7] dealing with the intersection of the ranges of two independent random walk, that we denote for any  $n \in \mathbb{N} \cup \{\infty\}$ ,  $\mathcal{R}_n = \{z : l_n(z) > 0\}$  and similarly for  $\tilde{\mathcal{R}}_n$ . Assume that  $d \geq 5$ , and for any  $\epsilon > 0$ , and  $L$  large enough

$$\exp\left(-L^{1-\frac{2}{d}+\epsilon}\right) \leq \mathbb{P}\left(|\mathcal{R}_\infty \cap \tilde{\mathcal{R}}_\infty| \geq L\right) \leq \exp\left(-L^{1-\frac{2}{d}-\epsilon}\right). \quad (1.10)$$

Finally, we remark that a large deviation principle has been established by Bolthausen, den Hollander and van den Berg in [4], for  $\{|\mathcal{R}_n \cap \tilde{\mathcal{R}}_n| \geq \xi n\}$  for large  $n$  and  $\xi > 0$ , or rather for the continuous counterpart, that is the intersection of the volume of two independent Wiener sausages. However, a large deviation principle for  $|\mathcal{R}_\infty \cap \tilde{\mathcal{R}}_\infty|$  is still open.

A lower bound for  $P(|\mathcal{L}(n) \cap \tilde{\mathcal{L}}(m)| \geq L)$  obtains following the strategy proposed in [7] in the proof of the lower bound for their Theorem 4. Thus, we force the random walk  $S_n$  (resp.  $\tilde{S}_n$ ) to make  $n \times L^{1-\frac{2}{d}+\epsilon}$ -returns to 0 (resp.  $m \times L^{1-\frac{2}{d}+\epsilon}$ -returns to 0), making probable that all sites of a ball of radius  $r$ , with  $|B(0, r)| \geq L \geq |B(0, r/2)|$ , are visited  $n$ -times (resp.  $m$ -times). Thus, the following lower bound holds.

**Corollary 1.9** *[Corollary of Proposition 6 of [7]] For any  $\epsilon > 0$ , and for  $L$  large enough, [7] prove that*

$$P\left(|\mathcal{L}(n) \cap \tilde{\mathcal{L}}(m)| \geq L\right) \geq \exp(-(n+m)L^{1-\frac{2}{d}+\epsilon}). \quad (1.11)$$

A direct consequence of Corollaries 1.7 and 1.9 is some information about the geometry of the intersection of two *high* level sets, knowing that the volume of the intersection is large. Thus, we formulate it in the following way.

**Corollary 1.10** *Let  $n_L$  and  $m_L$  such that  $\max(n_L, m_L) \gg L^{d/2}$  as  $L$  goes to infinity, then for any  $\epsilon > 0$*

$$\lim_{L \rightarrow \infty} P\left(\text{cap}\left(\mathcal{L}(n_L) \cap \tilde{\mathcal{L}}(m_L)\right) \geq L^{1-\frac{2}{d}+\epsilon} \mid |\mathcal{L}(n_L) \cap \tilde{\mathcal{L}}(m_L)| \geq L\right) = 0. \quad (1.12)$$

The paper is organized as follows. In Section 2, we prove Lemma 1.4, and then Proposition 1.3 as its corollary. We then show Proposition 1.1 in Section 3. Then, we focus on Proposition 1.5 in Section 4, which is technically the longest part of the paper. Finally, the proof of Corollary 1.7 follows in Section 5.

## 2 Proofs of Proposition 1.3 and of Lemma 1.4

### 2.1 Proof of Lemma 1.4

We assume  $d \geq 5$ . Lemma 1.4 can be thought of as an interpolation inequality between Lemma 1 and Lemma 2 of [7], whose proofs follow a classical pattern (in statistical physics) of estimating all moments of  $\zeta(q)$ . This control is possible since all quantities are expressed in terms of iterates of the Green's function, whose asymptotics are well known (see for instance Theorem 1.5.4 of [9]).

From [7], it is enough that for a positive constant  $C_q$ , we establish the following control on the moments

$$\forall n \in \mathbb{N}, \quad \mathbb{E}[\zeta(q)^n] \leq C_q^n (n!)^q. \quad (2.1)$$

First, noting that  $q - 1 \leq 1$ , we use Jensen's inequality in the last inequality

$$\begin{aligned} \mathbb{E}[\zeta(q)^n] &\leq \sum_{z_1, \dots, z_n \in \mathbb{Z}^d} E_0 \left[ \prod_{i=1}^n l_\infty(z_i) \right] E_0 \left[ \prod_{i=1}^n l_\infty(z_i)^{q-1} \right] \\ &\leq \sum_{z_1, \dots, z_n \in \mathbb{Z}^d} \left( E_0 \left[ \prod_{i=1}^n l_\infty(z_i) \right] \right)^q \end{aligned} \quad (2.2)$$

If  $\mathcal{S}_n$  is the set of permutation of  $\{1, \dots, n\}$  (with the convention that for  $\pi \in \mathcal{S}_n$ ,  $\pi(0) = 0$ ) we have,

$$\begin{aligned} E \left[ \prod_{i=1}^n l_\infty(z_i) \right] &= \sum_{s_1, \dots, s_n \in \mathbb{N}} P_0(S_{s_i} = z_i, \forall i = 1, \dots, n) \\ &\leq \sum_{\pi \in \mathcal{S}_n} \sum_{s_1 \leq s_2 \leq \dots \leq s_n \in \mathbb{N}} P_0(S_{s_i} = z_{\pi(i)}, \forall i = 1, \dots, n) \\ &\leq \sum_{\pi \in \mathcal{S}_n} \prod_{i=1}^n G_d(z_{\pi(i-1)}, z_{\pi(i)}). \end{aligned} \quad (2.3)$$

Now, by Hölder's inequality

$$\begin{aligned} \sum_{z_1, \dots, z_n} \left( \sum_{\pi \in \mathcal{S}_n} \prod_{i=1}^n G_d(z_{\pi(i-1)}, z_{\pi(i)}) \right)^q &\leq \sum_{z_1, \dots, z_n} (n!)^{q-1} \sum_{\pi \in \mathcal{S}_n} \prod_{i=1}^n G_d(z_{\pi(i-1)}, z_{\pi(i)})^q \\ &= (n!)^q \sum_{z_1, \dots, z_n} \prod_{i=1}^n G_d(z_{i-1}, z_i)^q \end{aligned} \quad (2.4)$$

Thus, classical estimates for the Green's function, (2.4) implies that

$$\begin{aligned} \sum_{z_1, \dots, z_n \in \mathbb{Z}^d} \left( E_0 \left[ \prod_{i=1}^n l_\infty(z_i) \right] \right)^q &\leq (n!)^q C^n \sum_{z_1, \dots, z_n} \prod_{i=1}^n (1 + \|z_i - z_{i-1}\|)^{q(2-d)} \\ &\leq (n!)^q C^n \left( \sum_{z \in \mathbb{Z}^d} (1 + \|z\|)^{d(2-d)} \right)^n. \end{aligned} \quad (2.5)$$

Thus, when  $d \geq 5$  and  $q > \frac{d}{d-2}$ , we have a constant  $C_q > 0$  such that

$$\mathbb{E}[\zeta(q)^n] \leq C_q^n (n!)^q \quad (2.6)$$

The Lemma follows now by routine consideration.

## 2.2 Proof of the Proposition 1.3

This follows easily from Lemma 1.4. Indeed, for  $q < 2$

$$\begin{aligned} \left\{ \sum_{z \notin \mathcal{D}(\frac{\sqrt{t}}{A}) \cap \tilde{\mathcal{D}}(\frac{\sqrt{t}}{A})} l_\infty(z) \tilde{l}_\infty(z) > t \right\} &\subset \left\{ \sum_{l_\infty(z) \leq \frac{\sqrt{t}}{A}} l_\infty(z)^{q-1} \tilde{l}_\infty(z) > \frac{t}{2} \left( \frac{A}{\sqrt{t}} \right)^{2-q} \right\} \\ &\cup \left\{ \sum_{\tilde{l}_\infty(z) \leq \frac{\sqrt{t}}{A}} l_\infty(z) \tilde{l}_\infty(z)^{q-1} > \frac{t}{2} \left( \frac{A}{\sqrt{t}} \right)^{2-q} \right\}. \end{aligned} \quad (2.7)$$

Then, since  $1 > \frac{2-q}{2}$ , Lemma 1.4 applied to (2.5) implies that for large  $t$

$$\mathbb{P} \left( \sum_{z \notin \mathcal{D}(\frac{\sqrt{t}}{A}) \cap \tilde{\mathcal{D}}(\frac{\sqrt{t}}{A})} l_\infty(z) \tilde{l}_\infty(z) > t \right) \leq \exp \left( -\kappa_d A^{\frac{2-q}{q}} t^{1/2} \right), \quad \text{since} \quad \frac{1}{q} \left( 1 - \frac{2-q}{2} \right) = \frac{1}{2}. \quad (2.8)$$

## 3 Proof of Proposition 1.1

The proof of Proposition 1.1 relies on [1], and we first recall some of its key steps we need here. Our main result in [1] (called there Theorem 1.1) is that there is a positive constant  $\mathcal{I}(2)$ , such that for  $\xi > 0$  the following limit exists

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \log P_0 (\langle l_n, l_n \rangle - E_0 [\langle l_n, l_n \rangle] > n\xi) = -\mathcal{I}(2) \sqrt{\xi}. \quad (3.1)$$

To prove (3.1), we observed that in order to produce an excess self-intersection local times, the walk spends a time of order  $\sqrt{n}$  over a finite number of sites. Thus, for some constant  $A$ , for any  $\epsilon > 0$ ,  $n$  large enough and a constant  $\gamma$  depending on  $A$

$$P_0 (\langle l_n, l_n \rangle - E_0 [\langle l_n, l_n \rangle] > n(1 + \epsilon)) \leq n^\gamma P_0 \left( \sum_{z \in \mathcal{D}_n(\sqrt{n}/A)} l_\infty^2(z) > n \right), \quad (3.2)$$

where  $\mathcal{D}_n(\xi) = \{z \in \mathbb{Z}^d : l_n(z) > \xi\}$ . Proposition 6.6 of [1] allows us to relocate  $\mathcal{D}_n(\sqrt{n}/A)$  into a finite region  $\Lambda_\epsilon$  of  $\mathbb{Z}^d$ , whose diameter is independent of  $n$ , so that for  $c > 0$

$$P \left( \sum_{\mathcal{D}_n(\frac{\sqrt{n}}{A})} l_\infty^2(z) > n \right) \leq e^{c\epsilon\sqrt{n}} P \left( \sum_{z \in \Lambda_\epsilon} l_\infty^2(z) > n, \Lambda_\epsilon \subset \mathcal{D}_\infty(\frac{\sqrt{n}}{A}), l_\infty(0) = \max_{\Lambda_\epsilon} l_\infty \right). \quad (3.3)$$

Then, visiting  $\sqrt{n}$ -times each site of  $\Lambda_\epsilon$  requires only a total time of order  $\sqrt{n}$ , and Proposition 7.1 of [1] yields an integer sequence  $\{k_n(z), z \in \Lambda_\epsilon\}$  with

$$\sum_{z \in \Lambda_\epsilon} k_n(z)^2 \geq n, \quad k_n(z) \leq A\sqrt{n},$$

and, constants  $\gamma, \alpha_0$  such that for  $\alpha > \alpha_0$

$$P \left( \sum_{z \in \Lambda_\epsilon} l_\infty^2(z) > n, \quad l_\infty(0) = \max_{\Lambda_\epsilon} l_\infty \right) \leq n^\gamma P \left( l_{\lfloor \alpha \sqrt{n} \rfloor} |_{\Lambda_\epsilon} = k_n |_{\Lambda_\epsilon}, \mathcal{B}(\lfloor \alpha \sqrt{n} \rfloor) \right), \quad (3.4)$$

where  $\mathcal{B}(m) = \{S_m = 0, l_m(0) = \max l_m(z)\}$ . On the other hand, note also the obvious lower bound: for  $\Lambda_\epsilon \subset \Lambda$

$$P \left( \sum_{\Lambda} l_\infty^2(z) > n \right) \geq P \left( l_{\lfloor \alpha \sqrt{n} \rfloor} |_{\Lambda_\epsilon} = k_n |_{\Lambda_\epsilon}, \mathcal{B}(\lfloor \alpha \sqrt{n} \rfloor) \right). \quad (3.5)$$

The subadditive argument treats the right hand side of (3.5), and yields also

$$\lim_{\Lambda \rightarrow \mathbb{Z}^d} \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \log P \left( \sum_{z \in \Lambda} l_\infty^2(z) > n \right) = -\mathcal{I}(2). \quad (3.6)$$

Now, we proceed with the link with intersection local times. First, as mentioned in Remark 1.2, Chen and Mörters prove also that for any finite  $\Lambda \subset \mathbb{Z}^d$

$$\lim_{n \rightarrow \infty} \frac{1}{n^{1/2}} \log P \left( \left\langle \mathbb{1}_\Lambda l_\infty, \tilde{l}_\infty \right\rangle > n \right) = -2I_{CM}(\Lambda),$$

with  $I_{CM}(\Lambda)$  converging to  $I_{CM}$  as  $\Lambda$  increases to cover  $\mathbb{Z}^d$ . The important feature is that for any fixed  $\epsilon > 0$ , we can fix a finite  $\Lambda$  subset of  $\mathbb{Z}^d$  such that  $|I_{CM}(\Lambda) - I_{CM}| \leq \epsilon$ . Note now that by Cauchy-Schwarz' inequality, and for finite set  $\Lambda$

$$\sum_{z \in \Lambda} l_\infty(z) \tilde{l}_\infty(z) \leq \left( \sum_{\Lambda} l_\infty^2(z) \right)^{\frac{1}{2}} \left( \sum_{\Lambda} \tilde{l}_\infty^2(z) \right)^{\frac{1}{2}}. \quad (3.7)$$

Inequalities (3.6) and (3.7) imply by routine consideration that

$$\limsup_{\Lambda \rightarrow \mathbb{Z}^d} \limsup_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \log P \left( \left\langle \mathbb{1}_\Lambda l_\infty, \tilde{l}_\infty \right\rangle > n \right) \leq -\mathcal{I}(2) \inf_{\alpha > 0} \left\{ \sqrt{\alpha} + \frac{1}{\sqrt{\alpha}} \right\} = -2\mathcal{I}(2). \quad (3.8)$$

Note also the obvious lower bound for  $\Lambda_\epsilon \subset \Lambda$

$$P \left( \sum_{z \in \Lambda} l_\infty(z) \tilde{l}_\infty(z) > n \right) \geq P \left( l_{\lfloor \alpha \sqrt{n} \rfloor} |_{\Lambda_\epsilon} = k_n |_{\Lambda_\epsilon}, \mathcal{B}(\lfloor \alpha \sqrt{n} \rfloor) \right)^2. \quad (3.9)$$

Since  $\epsilon$  is arbitrary, we obtain

$$\liminf_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \log P \left( \left\langle l_\infty, \tilde{l}_\infty \right\rangle > n \right) \geq -2\mathcal{I}(2). \quad (3.10)$$

(3.8) and (3.10) conclude the proof of Proposition 1.1.

## 4 Proof of Proposition 1.5

Site 0 of  $\mathbb{Z}^d$  plays a special rôle, since the walk start from 0. We assume that  $0 \notin \Lambda$ , and omit to write the trivial changes brought by the case  $0 \in \Lambda$ .

Our first step is to decompose paths in  $\{l_\infty(z) = n(z), \forall z \in \Lambda\}$  in terms of their sequence of crossed oriented edges. Let  $t = \sum_{z \in \Lambda} n(z)$ , the total time the walk spends on  $\Lambda$ , and let our state space be

$$\Omega_t = \{\sigma \in (\{0\} \times \Lambda) \times (\Lambda \times \Lambda)^{t-1} \times (\Lambda \times \{\infty\}) : \sigma_2(i) = \sigma_1(i+1), \forall i = 1, \dots, t\}.$$

For an (oriented) edge  $e \in \Lambda \times \Lambda$  (resp. for a site  $z \in \Lambda$ ), we denote by  $l_\infty(e; \cdot)$  (resp.  $l_\infty(z; \cdot)$ ) the variable on  $\Omega_t$  counting the crossings of  $e$  (resp. the visits of  $z$ ). Thus

$$l_\infty(e; \sigma) = \sum_{s=1}^t \mathbb{I}\{\sigma(s) = e\}, \quad \text{and} \quad l_\infty(z; \sigma) = \sum_{s=1}^t \mathbb{I}\{\sigma_2(s) = z\}. \quad (4.1)$$

Now, to write concisely the decomposition we alluded to, we introduce the following handy notation: for an edge  $e = (x, y)$ , we set  $q(e) = q(x, y) = P_x(T_\Lambda < \infty, S_{T_\Lambda} = y)$ ,

$$P_0(l_\infty(z) = n(z), \forall z \in \Lambda) = \sum_{\sigma \in \Omega_t} \mathbb{I}\{l_\infty(z; \sigma) = n(z), \forall z \in \Lambda\} \left( \prod_{i=1}^t q(\sigma(i)) \right) P_{\sigma_2(t)}(T_\Lambda = \infty). \quad (4.2)$$

Now, we interpret  $\sigma \in \Omega_t$  by specifying the fate of each site of  $\Lambda$ . We associate, to each  $z \in \Lambda$ , a vector with  $n(z)$  entries: the  $k$ -th entry is the endpoint of the  $k$ -th visited edge with starting vertex  $z$  (and if  $z$  is the last visited site of  $\Lambda$ , then the  $n(z)$ -entry is  $\infty$ ). To identify fully the circuit, we need to specify  $S_{T_\Lambda}$ , the first visited site of  $\Lambda$  which we associate with 0. The family of vectors (with the 1-vector associate with 0) is equivalent to  $\sigma$ , and specify a circuit over  $\Lambda$ . Note that for any integer sequence  $\{\mathcal{E}(e), e \in \Lambda^2\}$ , we have the multinomial domination

$$|\{\sigma \in \Omega_t : l_\infty(e; \sigma) = \mathcal{E}(e), \forall e \in \Lambda^2\}| \leq \prod_{z \in \Lambda} \frac{(\sum_{x \in \Lambda} \mathcal{E}(z, x))!}{\prod_{x \in \Lambda} \mathcal{E}(z, x)!}. \quad (4.3)$$

Since the event of interest is given in terms of visits of sites of  $\Lambda$ , we need to express  $l_\infty(z; \sigma)$  in terms of  $\{l_\infty(e; \sigma), e \in \Lambda^2\}$ . The number of visits of  $z \in \Lambda$  is essentially the total crossings of all edges with initial vertex  $z$ . That is, for  $\sigma \in \Omega_t$

$$l_\infty(z; \sigma) = \sum_{x \in \Lambda} l_\infty((z, x); \sigma) + \mathbb{I}\{z = \sigma_2(t)\}. \quad (4.4)$$

For  $z_1, z_t \in \Lambda$ , let  $\mathcal{H}(z_1, z_t)$  be the images of  $\{l_\infty(e, \sigma), e \in \Lambda^2\}$  as  $\sigma$  spans  $\Omega_t$  with  $\sigma_2(1) = z_1, \sigma_2(t) = z_t$ , under the additional constraint that for  $\mathcal{E} \in \mathcal{H}(z_1, z_t)$ , and all  $z \in \Lambda$ ,  $\sum_{x \in \Lambda} \mathcal{E}(z, x) = n(z) - 1\{z = z_t\}$ .

We can now express (4.2) into a sum over edge crossings. We need however to distinguish the first and last points in  $\Lambda$ , say  $z_1$  and  $z_t$  respectively. Thus,

$$P_0(l_\infty(z) = n(z), \forall z \in \Lambda) \leq \sum_{z_1, z_t \in \Lambda} q(0, z_1) \sum_{\mathcal{E} \in \mathcal{H}(z_1, z_t)} \prod_{e \in \Lambda^2} q(e)^{\mathcal{E}(e)} \sum_{\sigma \in \Omega_t} \mathbb{I}\{l_\infty(\bullet; \sigma) = \mathcal{E}(\bullet)\}$$



$$\leq \sum_{z_1, z_t \in \Lambda} q(0, z_1) \sum_{\mathcal{E} \in \mathcal{H}(z_1, z_t)} \prod_{e \in \Lambda^2} q(e)^{\mathcal{E}(e)} \prod_{z \in \Lambda} \frac{(\sum_{x \in \Lambda} \mathcal{E}(z, x))!}{\prod_{x \in \Lambda} \mathcal{E}(z, x)!}. \quad (4.5)$$

The proof proceeds now in two steps which we have gathered in two distinct sections for the ease of reading.

Step 1: Given  $z_1, z_t \in \Lambda$  and the edge-occupation trace of path  $\mathcal{E} = \{\mathcal{E}(e), e \in \Lambda^2\} \in \mathcal{H}(z_1, z_t)$ , we wish to extract a path visiting each vertex of  $\Lambda$  at least once and whose total (graph) length grows like  $|\Lambda|$  (and not  $|\Lambda|^2$ ). We overcome this difficulty by building a self-avoiding path covering  $\Lambda$  whose set of crossed-edges is not a subset of  $\mathcal{E}$ , but for which we have some domination of the product over edge-distances (see (4.6)). This is done in Section 4.1.

Step 2: Extract the contribution of the self-avoiding path in (4.5), and re-adjust the sum in (4.5) so that a true multinomial distribution appears, as well as the capacity of  $\Lambda$ . This is done in Section 4.2.

## 4.1 Extracting a Self-Avoiding path

We fix  $z_1, z_t \in \Lambda$ , and  $\mathcal{E} \in \mathcal{H}(z_1, z_t)$ . Let  $\gamma$  be a path giving rise to the edge-occupation numbers  $\mathcal{E}$ . Our aim in this section is to extract a self-avoiding path over  $\Lambda$  which *uses* the edges of  $\mathcal{E}$ . The self-avoiding path we eventually build is not a sub-path of  $\gamma$ , though it starts with  $z_1$ . To describe in details the construction, we first set notations. A self-avoiding path over  $\Lambda$ , which we call for simplicity a  $\Lambda$ -*trail*, is an element of  $(\Lambda^2)^{|\Lambda|-1}$ , say  $\{e_1, \dots, e_{|\Lambda|-1}\}$  such that for  $i = 1, \dots, |\Lambda| - 2$ , the end-vertex of  $e_i$  is the starting-vertex of  $e_{i+1}$ , and so that no vertex is used twice. The set of loops of  $\Lambda$  is denoted  $\Delta_\Lambda = \{(z, z) : z \in \Lambda\}$ . Now, for each  $z_1, z_t \in \Lambda$ , and  $\mathcal{E} \in \mathcal{H}(z_1, z_t)$ , we call a  $\mathcal{E}$ -stock, an element  $\mathcal{S} \in \mathbb{N}^{\Lambda \times \Lambda}$  with  $\mathcal{S} \leq \mathcal{E}$  for the natural order in  $\mathbb{N}^{\Lambda \times \Lambda}$ , with  $\mathcal{S}|_\Delta \equiv 0$ ,  $\sum_{e \in \Lambda \times \Lambda} \mathcal{S}(e) = |\Lambda| - 1$ , and such that for each vertex  $z \in \Lambda$ , there is at most one oriented edge  $e \in \{z\} \times \Lambda$  such that  $\mathcal{S}(e) > 0$ . Let us denote by  $(\mathcal{S})$  this latter property. Note that contrary to the edge-occupation number  $\mathcal{E}$ , a  $\mathcal{E}$ -stock may not correspond to a path.

We show in this section that for each  $z_1, z_t \in \Lambda$ , and  $\mathcal{E} \in \mathcal{H}(z_1, z_t)$ , there is a  $\Lambda$ -trail  $\mathcal{T}$ , with initial vertex  $z_1$ , and a  $\mathcal{E}$ -stock  $\mathcal{S}$  such that if for edge  $e = (z, z')$  we denote by  $d(e) = \|z - z'\|$ , then

$$|\Lambda|! \prod_{e \in \Lambda^2} d(e)^{\mathcal{S}(e)} \geq \prod_{e \in \mathcal{T}} d(e). \quad (4.6)$$

### 4.1.1 Coalescing $\Lambda$

Let  $N = |\Lambda|$ , and call  $\Lambda_N = \Lambda$  to emphasize that  $\Lambda$  contains  $N$  vertices. We coalesce  $\Lambda$  through a sequence of graphs  $\Lambda_{N-1}, \dots, \Lambda_1$  such that for  $k = 1, \dots, N$  the graph  $\Lambda_k$  has exactly  $k$  vertices. Then, we show (4.6) by induction on  $\Lambda_k$ . We first describe how to obtain  $\Lambda_{N-1}$ . Choose a pair of vertices, say  $e_N = (a_N, \tilde{a}_N)$  of  $\Lambda$  which minimize the euclidean distance.  $\Lambda_{N-1}$  is obtained from  $\Lambda_N$  by suppressing  $a_N, \tilde{a}_N$  but adding one new vertex,

say  $\psi_{N-1}$ , which we also think of as a cluster containing  $a_N$  and  $\tilde{a}_N$ . We also define a pseudo-metric  $d_{N-1} : \Lambda_{N-1}^2 \rightarrow \mathbb{R}^+$  by

$$\forall z, z' \in \Lambda_{N-1} \setminus \{\psi_{N-1}\}, \quad d_{N-1}(z, z') = d(z, z'),$$

and

$$d_{N-1}(z, \psi_{N-1}) = d_{N-1}(\psi_{N-1}, z) = \min(d(z, a_N), d(z, \tilde{a}_N)).$$

Note that  $d_{N-1}$  fails to satisfy the triangle inequality. Now, we can associate with  $\gamma$  and  $\Lambda_{N-1}$  an edge-occupation number  $\mathcal{E}_{N-1}$ . In other words, from path  $\gamma$ , we count edge-crossings as if  $a_N$  and  $\tilde{a}_N$  were indistinguishable, and obtain a path on  $\Lambda_{N-1}$  which we call  $\gamma_{N-1}$ . Thus, for  $z \in \Lambda_{N-1} \setminus \{\psi_{N-1}\}$

$$\mathcal{E}_{N-1}(z, \psi_{N-1}) = \mathcal{E}(z, a_N) + \mathcal{E}(z, \tilde{a}_N), \quad \mathcal{E}_{N-1}(\psi_{N-1}, z) = \mathcal{E}(a_N, z) + \mathcal{E}(\tilde{a}_N, z), \quad (4.7)$$

and,  $\mathcal{E}_{N-1}(\psi_{N-1}, \psi_{N-1}) = \mathcal{E}(a_N, \tilde{a}_N) + \mathcal{E}(\tilde{a}_N, a_N) + \mathcal{E}(a_N, a_N) + \mathcal{E}(\tilde{a}_N, \tilde{a}_N)$ . We proceed by induction until we reach  $\Lambda_1$  with one vertex, which we think of as a cluster of  $N$  vertices of  $\Lambda$ .

Note that for  $k = 2, \dots, N$ , and  $z, z' \in \Lambda_k$ , we keep in mind that  $z, z'$  can be thought of as clusters in  $\Lambda_N$ , and that

$$d_k(z, z') = \min \{d(x, x') : x, x' \in \Lambda_N, x \in z, x' \in z'\}. \quad (4.8)$$

Also, we call  $\gamma_k$  the path obtained from  $\gamma$  on  $\Lambda_k$  by identifying all vertices (of  $\Lambda$ ) belonging to a single cluster (i.e. a vertex of  $\Lambda_k$ ).

#### 4.1.2 Proof of (4.6) by induction

We want that  $z_1$  be the starting point of the trail. Note that  $\Lambda_2$  has two vertices  $\{a_2, \tilde{a}_2\}$ , and since all sites of  $\Lambda$  are visited, we have necessarily  $\mathcal{E}_2(a_2, \tilde{a}_2) > 0$  when  $z_1 \in a_2$  and  $\mathcal{E}_2(\tilde{a}_2, a_2) > 0$  when  $z_1 \in \tilde{a}_2$ , so we can define an  $\mathcal{E}_2$ -block  $\mathcal{S}_2(a_2, \tilde{a}_2) = 1$  (resp.  $\mathcal{S}_2(\tilde{a}_2, a_2) = 1$ ), and the  $\Lambda_2$ -trail  $\mathcal{T}_2 = \{(a_2, \tilde{a}_2)\}$  (resp.  $\mathcal{T}_2 = \{(\tilde{a}_2, a_2)\}$ ), when  $z_1$  belongs to the first (resp. second) cluster.

Assume now that we have an  $\mathcal{E}_{k-1}$ -stock  $\mathcal{S}_{k-1}$ , and a  $\Lambda_{k-1}$ -trail  $\mathcal{T}_{k-1}$  so that

$$\frac{N!}{(N - (k - 2))!} \prod_{e \in \Lambda_{k-1}^2} d_{k-1}(e)^{\mathcal{S}_{k-1}(e)} \geq \prod_{e \in \mathcal{T}_{k-1}} d_{k-1}(e). \quad (4.9)$$

Recall that  $e_k = (a_k, \tilde{a}_k)$  has been coalesced to produce vertex  $\psi_{k-1}$  of  $\Lambda_{k-1}$ . Note that trail  $\mathcal{T}_{k-1}$  crosses  $\psi_{k-1}$  only once. Also, it is part of our induction hypothesis to assume that the first crossed vertex of  $\mathcal{T}_{k-1}$  when seen as a cluster, contains  $z_1$ .

Thus, assume first that  $\psi_{k-1}$  is not the first vertex of the trail  $\mathcal{T}_{k-1}$ . Let  $b, b' \in \Lambda_{k-1}$  such that  $(b, \psi_{k-1})$  and  $(\psi_{k-1}, b') \in \mathcal{T}_{k-1}$ .

By construction,  $\mathcal{E}_k$  satisfies for any  $z \in \Lambda_k$

$$\mathcal{E}_k(z, a_k) + \mathcal{E}_k(z, \tilde{a}_k) = \mathcal{E}_{k-1}(z, \psi_{k-1}) \geq \mathcal{S}_{k-1}(z, \psi_{k-1}), \quad (4.10)$$

and,

$$\mathcal{E}_k(a_k, z) + \mathcal{E}_k(\tilde{a}_k, z) = \mathcal{E}_{k-1}(\psi_{k-1}, z) \geq \mathcal{S}_{k-1}(\psi_{k-1}, z).$$

Thus, we can define  $\tilde{\mathcal{S}}_k$  on  $\Lambda_k^2$  such that for  $z \in \Lambda_k \cap \Lambda_{k-1}$

$$\tilde{\mathcal{S}}_k(z, a_k) + \tilde{\mathcal{S}}_k(z, \tilde{a}_k) = \mathcal{S}_{k-1}(z, \psi_{k-1}), \quad \tilde{\mathcal{S}}_k(z, a_k) \leq \mathcal{E}_k(z, a_k), \quad \text{and} \quad \tilde{\mathcal{S}}_k(z, \tilde{a}_k) \leq \mathcal{E}_k(z, \tilde{a}_k).$$

and similarly for  $\tilde{\mathcal{S}}_k(a_k, z)$  and  $\tilde{\mathcal{S}}_k(\tilde{a}_k, z)$ . Also, for  $z, z' \in \Lambda_k \setminus \{a_k, \tilde{a}_k\}$ ,  $\tilde{\mathcal{S}}_k(z, z') = \mathcal{S}_{k-1}(z, z')$ .

Note that in path  $\gamma_k$ , there is at least one crossing of each vertex of  $\Lambda_k$ . Thus,

$$\sum_{e \in \Lambda_k \setminus \Delta_k} \mathcal{E}_k(e) \geq k - 1, \quad \text{where} \quad \Delta_k := \Delta_{\Lambda_k}. \quad (4.11)$$

Now, by the induction hypothesis,  $\mathcal{S}_{k-1}$  satisfies  $\sum_e \mathcal{S}_{k-1}(e) = k - 2$  and  $(\mathcal{S})$ . The simple observation which guarantees property  $(\mathcal{S})$  to propagate through induction is that there cannot be two vertices of  $\Lambda_k$ , say  $x$  and  $y$ , such that

$$\forall z \in \Lambda_k \setminus \{x\} \quad \mathcal{E}_k(x, z) = 0, \quad \text{and} \quad \forall z \in \Lambda_k \setminus \{y\} \quad \mathcal{E}_k(y, z) = 0.$$

The reason is that when looking at  $\gamma_k$ , at most only one of  $x$  or  $y$  can be the last visit in  $\Lambda_k$  (and in this case, the last vertex points to no other vertex of  $\Lambda_k$ ). We consider now two cases. Assume first, that

$$\forall z \in \Lambda_k \setminus \{a_k\} \quad \tilde{\mathcal{S}}_k(a_k, z) = 0, \quad \text{and} \quad \forall z \in \Lambda_k \setminus \{\tilde{a}_k\} \quad \tilde{\mathcal{S}}_k(\tilde{a}_k, z) = 0.$$

Then, we just saw that there is  $z^* \in \Lambda_k$  such that  $\mathcal{E}_k(a_k, z^*) > 0$  and  $z^* \neq a_k$ , or  $\mathcal{E}_k(\tilde{a}_k, z^*) > 0$  and  $z^* \neq \tilde{a}_k$ . In the case  $\mathcal{E}_k(a_k, z^*) > 0$  (resp.  $\mathcal{E}_k(\tilde{a}_k, z^*) > 0$ ), we choose  $e_k^* = (a_k, z^*)$  (resp.  $e_k^* = (\tilde{a}_k, z^*)$ ), and  $\mathcal{S}_k = \tilde{\mathcal{S}}_k + 1_{e_k^*}$ . Secondly, assume that for  $z^* \in \Lambda_k$ , we have  $\tilde{\mathcal{S}}_k(a_k, z^*) > 0$  and necessarily  $\tilde{\mathcal{S}}_k(\tilde{a}_k, z) = 0$  for all  $z \in \Lambda_k$  (the other case  $\tilde{\mathcal{S}}_k(\tilde{a}_k, z^*) > 0$  is similar). By the induction hypothesis, there is another vertex of  $\Lambda_k$ , say  $a^*$  such that for any  $z \in \Lambda_k$ , we have  $\tilde{\mathcal{S}}_k(a^*, z) = 0$ . We then proceed as in the first case to deduce that there is an edge  $e_k^*$  incident to either  $\tilde{a}_k$  or  $a^*$  such that  $\mathcal{E}_k(e_k^*) > 0$ , and  $e_k^*$  is not a loop.

Thus, there is an edge  $e_k^* \in \Lambda_k^2 \setminus \Delta_k$  such that  $\mathcal{E}_k(e_k^*) > \tilde{\mathcal{S}}_k(e_k^*)$ , where  $\tilde{\mathcal{S}}_k$  is built from  $\mathcal{S}_{k-1}$  as in the previous case. We thus set  $\mathcal{S}_k = \tilde{\mathcal{S}}_k + 1_{e_k^*}$ , and note that by definition

$$d_k(a_k, \tilde{a}_k) \leq d_k(e_k^*). \quad (4.12)$$

For the ease of reading we distinguish two cases.

Case 1:  $d_{k-1}(b, \psi_{k-1}) = d_k(b, a_k)$  and  $d_{k-1}(\psi_{k-1}, b') = d_k(\tilde{a}_k, b')$

The trail  $\mathcal{T}_k$  is the same as trail  $\mathcal{T}_{k-1}$  but with edges  $\{(b, \psi_{k-1}), (\psi_{k-1}, b')\}$  replaced by edges  $\{(b, a_k), (a_k, \tilde{a}_k), (\tilde{a}_k, b')\}$ . Thus, we have

$$d_k(e_k^*) \times d_{k-1}(b, \psi_{k-1})d_{k-1}(\psi_{k-1}, b') \geq d_k(b, a_k)d_k(a_k, \tilde{a}_k)d_k(\tilde{a}_k, b'). \quad (4.13)$$

The same reasoning (with obvious changes of symbols) would hold if  $d_{k-1}(b, \psi_{k-1}) = d_k(\tilde{a}_k, b)$  and  $d_{k-1}(\psi_{k-1}, b') = d_k(a_k, b')$ .

Case 2:  $d_{k-1}(b, \psi_{k-1}) = d_k(b, a_k)$  ,and  $d_{k-1}(\psi_{k-1}, b') = d_k(a_k, b')$ .

We have to think of  $a_k, \tilde{a}_k, b'$  as clusters, and let  $x, x' \in a_k$ ,  $y, y' \in \tilde{a}_k$  and  $z, z' \in b'$  be vertices of  $\Lambda_N$  such that

$$d(x', z) = d_k(a_k, b'), \quad d(x, y') = d_k(a_k, \tilde{a}_k), \quad \text{and} \quad d(y, z') = d_k(\tilde{a}_k, b'). \quad (4.14)$$

Since  $x, x'$  belong to the same cluster, our construction implies that there is a sequence  $\{x_0, x_1, \dots, x_{k_1}\}$  vertices of  $a_k$  with

$$x_0 = x, \quad x_{k_1} = x', \quad \text{and} \quad \forall i = 1, \dots, k_1 \quad d(x_{i-1}, x_i) \leq d_k(a_k, \tilde{a}_k). \quad (4.15)$$

Similarly, we consider  $\{y_0, y_1, \dots, y_{k_2}\}$  in  $\tilde{a}_k$  joining  $y$  and  $y'$  and  $\{z_0, z_1, \dots, z_{k_3}\}$  in  $b'$  joining  $z$  and  $z'$ . The important observation is that  $k_1 + k_2 + k_3 \leq N - k$ . Thus, by the triangle inequality

$$\begin{aligned} d_k(\tilde{a}_k, b') = d(y, z') &\leq \sum_{i=1}^{k_2} d(y_{i-1}, y_i) + d(y', x) + \sum_{i=1}^{k_1} d(x_{i-1}, x_i) + d(x', z) + \sum_{i=1}^{k_3} d(z_{i-1}, z_i) \\ &\leq (N - k + 1)d_k(a_k, \tilde{a}_k) + d_k(a_k, b') \\ &\leq (N - k + 2)d_k(a_k, b') \end{aligned} \quad (4.16)$$

Using (4.16), we have

$$(N - k + 2)d_k(e_k^*)d_{k-1}(b, \psi_{k-1})d_{k-1}(\psi_{k-1}, b') \geq d_k(b, a_k)d_k(a_k, \tilde{a}_k)d_k(\tilde{a}_k, b'). \quad (4.17)$$

Thus, with the same trail as in the previous case, we have from (4.9) and (4.17)

$$\frac{N!}{(N - k + 1)!} \prod_{e \in \Lambda_k^2} d_k(e)^{S_k(e)} \geq \prod_{e \in \mathcal{T}_k} d_k(e). \quad (4.18)$$

Finally, we explain how to manage so that  $z_1$  remains in the first cluster of  $\mathcal{T}_k$ . Assume that  $\psi_{k-1}$  contains  $z_1$  (and necessarily  $\psi_{k-1}$  is the first vertex of  $\mathcal{T}_{k-1}$ ). If  $z_1 \in a_k$ , then add a fictitious vertex  $b^\dagger$  in  $\Lambda_k$ , and an edge  $(b^\dagger, a_k)$ , and proceed as before with  $b^\dagger$  in the rôle of  $b$ . However, if  $z_1 \in \tilde{a}_k$ , the fictitious edge is  $(b^\dagger, \tilde{a}_k)$ , and  $\tilde{a}_k$  replaces  $a_k$  in the previous constructions.

## 4.2 About Outer Capacity

For  $z_1, z_t \in \Lambda$  and  $\mathcal{E} \in \mathcal{H}(z_1, z_t)$ , let  $\mathcal{S}$  be the  $\mathcal{E}$ -stock built in Section 4.1. The first step is to subtract edges of  $\mathcal{S}$  from  $\mathcal{E}$ , and replace them with loops of  $\Delta$ . Also, the number of edges of  $\mathcal{S}$  incident with  $z \in \Lambda$ , is denoted  $\text{leaf}(z, \mathcal{S})$ , that is  $\text{leaf}(z, \mathcal{S}) = \sum_{z'} \mathcal{S}(z, z') \in \{0, 1\}$ . Thus, we define for  $e \notin \Delta$

$$\mathcal{E}_\mathcal{S}(e) = \mathcal{E}(e) - \mathcal{S}(e) \geq 0, \quad (4.19)$$

and,

$$\mathcal{E}_\mathcal{S}(z, z) = \mathcal{E}(z, z) + \text{leaf}(z, \mathcal{S}) \quad (\text{recall that } \mathcal{S}(z, z) = 0). \quad (4.20)$$

Note that we might have  $\mathcal{E}_S \notin \mathcal{H}(z_1, z_t)$ , but at least each vertex  $z$  of  $\Lambda$  occurs  $n(z)$  times, in the sense that

$$\forall z \in \Lambda, \quad \sum_{z' \in \Lambda} \mathcal{E}_S(z, z') = \sum_{z' \in \Lambda} \mathcal{E}(z, z'). \quad (4.21)$$

The important (and simple) observation is that for  $\mathcal{E} \in \mathcal{H}(z_1, z_t)$

$$\prod_{z \in \Lambda} \frac{(\sum_{x \in \Lambda} \mathcal{E}(z, x))!}{\prod_{x \in \Lambda} \mathcal{E}(z, x)!} \leq \left( \max_{z \in \Lambda} n(z) \right)^{|\Lambda|} \prod_{z \in \Lambda} \frac{(\sum_{x \in \Lambda} \mathcal{E}_S(z, x))!}{\prod_{x \in \Lambda} \mathcal{E}_S(z, x)!} \quad (4.22)$$

Indeed, in view of (4.21), (4.22) is equivalent to showing that

$$\prod_{e \in \Delta_\Lambda} \frac{\mathcal{E}_S(e)!}{\mathcal{E}(e)!} \leq \bar{n}^{|\Lambda|} \prod_{e \notin \Delta_\Lambda} \frac{\mathcal{E}(e)!}{\mathcal{E}_S(e)!}. \quad (4.23)$$

Now since  $\mathcal{E}(e) \geq \mathcal{E}_S(e)$  for  $e \notin \Delta_\Lambda$ , Now, (4.23) would follow from

$$\prod_{z \in \Lambda} \frac{(\mathcal{E}(z, z) + \text{leaf}(z, \mathcal{S}))!}{\mathcal{E}(z, z)!} \leq \bar{n}^{|\Lambda|}.$$

Note that  $\mathcal{E}(z, z) + \text{leaf}(z, \mathcal{S}) \leq \sum_x \mathcal{E}(z, x) \leq n(z)$ , so that

$$\frac{(\mathcal{E}(z, z) + \text{leaf}(z, \mathcal{S}))!}{\mathcal{E}(z, z)!} \leq n(z)^{\text{leaf}(z, \mathcal{S})},$$

and since  $\sum_z \text{leaf}(z, \mathcal{S}) = \sum_e \mathcal{S}(e) = |\Lambda| - 1$ , we have for  $e \in \Delta_\Lambda$

$$\prod_{z \in \Lambda} \frac{(\mathcal{E}(z, z) + \text{leaf}(z, \mathcal{S}))!}{\mathcal{E}(z, z)!} \leq \prod_{z \in \Lambda} n(z)^{\text{leaf}(z, \mathcal{S})} \leq (\bar{n})^{|\Lambda|}. \quad (4.24)$$

Also, since the walk has probability  $\frac{1}{2d+1}$  to stay still, we have

$$q(e) \geq \frac{1}{2d+1}, \quad (4.25)$$

so that using that  $\sum_e \mathcal{S}(e) = |\Lambda| - 1$

$$\prod_{e \in \Lambda^2} q(e)^{\mathcal{E}(e) - \mathcal{S}(e)} \leq (2d+1)^{|\Lambda|} \prod_{e \in \Lambda^2} q(e)^{\mathcal{E}_S(e)}. \quad (4.26)$$

Now, we call  $\mathbb{T}(z_1)$  the set of  $\Lambda$ -trails with initial vertex  $z_1$ , and combine (4.22), (4.26) into (4.5) to obtain

$$\begin{aligned} P_0(l_\infty(z) = n(z), \forall z \in \Lambda) &\leq ((2d+1)\bar{n})^{|\Lambda|} \sum_{z_1, z_t \in \Lambda} q(0, z_1) \\ &\quad \sum_{\mathcal{T} \in \mathbb{T}(z_1)} \sum_{\mathcal{E} \leftrightarrow \mathcal{T}} \prod_{e \in \Lambda^2} q(e)^{\mathcal{S}(e)} \prod_{e \in \Lambda^2} q(e)^{\mathcal{E}_S(e)} \prod_{z \in \Lambda} \frac{(\sum_{x \in \Lambda} \mathcal{E}_S(z, x))!}{\prod_{x \in \Lambda} \mathcal{E}_S(z, x)!}. \end{aligned} \quad (4.27)$$

By  $\mathcal{E} \leftrightarrow \mathcal{T}$  we mean that  $\mathcal{E} \in \mathcal{H}(z_1, z_t)$  gives rise to trail  $\mathcal{T}$ , and by  $\mathcal{S}$  we denote the  $\mathcal{E}$ -stock associated with  $\mathcal{E}$ .

We make three observations.

1. For any fixed  $z \in \Lambda$ ,

$$\prod_{x \in \Lambda} P_z(S_{T_\Lambda} = x)^{\mathcal{E}(z,x)} = P_z(T_\Lambda < \infty)^{\sum_x \mathcal{E}(z,x)} \times \prod_{x \in \Lambda} p(z, x)^{\mathcal{E}(z,x)}, \quad (4.28)$$

where, for  $x, z \in \Lambda$  we defined

$$p(z, x) = P_z(S_{T_\Lambda} = x | T_\Lambda < \infty), \quad \text{wich satisfy} \quad \sum_{x \in \Lambda} p(z, x) = 1. \quad (4.29)$$

Also, using that  $1 - x \leq \exp(-x)$  (and with  $\underline{n} = \min(n(z))$ )

$$\begin{aligned} \prod_{z \in \Lambda} P_z(T_\Lambda < \infty)^{n(z)-1\{z=z_t\}} &\leq \exp \left( - \sum_{z \in \Lambda} P_z(T_\Lambda = \infty)(n(z) - 1\{z = z_t\}) \right) \\ &\leq \exp \left( 1 - \underline{n} \sum_{z \in \Lambda} P_z(T_\Lambda = \infty) \right) = e^{1-\underline{n} \text{cap}(\Lambda)}, \end{aligned} \quad (4.30)$$

where  $\text{cap}(\Lambda)$  denotes the capacity of the finite set  $\Lambda$ , given by  $\text{cap}(\Lambda) = \sum_{z \in \Lambda} P_z(T_\Lambda = \infty)$  (see Section 2.2 of [9]).

2. For any edge  $e$  joining  $z_1, z_2 \in \Lambda$

$$q(e) = P_{z_1}(S_{T_\Lambda} = z_2) \leq P_{z_1}(H(z_2) < \infty), \quad (4.31)$$

and well known asymptotics (see e.g. Theorem 1.5.4 of [9]), say that for  $d \geq 3$ , there are positive constants  $c_d, \underline{c}_d$  such that for any  $z_1$  and  $z_2$  distinct vertices of  $\mathbb{Z}^d$

$$\underline{c}_d \|z_1 - z_2\|^{2-d} \leq P_{z_1}(H(z_2) < \infty) \leq c_d \|z_1 - z_2\|^{2-d}. \quad (4.32)$$

Thus, recalling that for  $e = (z_1, z_2)$  we set  $d(e) := \|z_1 - z_2\|$ , inequality (4.6) of Section 4.1 yields for  $\mathcal{E}$ -block  $\mathcal{S}$  and  $\Lambda$ -trail  $\mathcal{T}$

$$\prod_{e \in \Lambda^2} q(e)^{\mathcal{S}(e)} \leq c_d^{|\Lambda|-1} \left( \prod_{e \in \Lambda^2} d(e)^{\mathcal{S}(e)} \right)^{2-d} \leq c_d^{|\Lambda|-1} (|\Lambda|!)^{d-2} \left( \prod_{e \in \mathcal{T}} d(e) \right)^{2-d}. \quad (4.33)$$

3. Because of property  $(\mathcal{S})$ , the transformation  $\mathcal{E} \rightarrow \mathcal{E}_{\mathcal{S}}$  can send at most  $(|\Lambda| - 1)^{|\Lambda|}$  edge-crossing configurations to the same image. Indeed, for each vertex we have  $|\Lambda| - 1$  possible edges (to the  $|\Lambda| - 1$  distinct vertices) which could have been changed into a self-loop. Also,  $(|\Lambda| - 1)^{|\Lambda|} \leq e^{|\Lambda|} |\Lambda|!$ .

Thus, with  $c'_d = 2e^2 c_d (2d + 1)$ , (4.27) becomes

$$\begin{aligned} P_0(l_\infty(z) = n(z), \forall z \in \Lambda) &\leq (c'_d \bar{n})^{|\Lambda|} (|\Lambda|!)^{d-2} e^{-\underline{n} \text{cap}(\Lambda)} \sum_{z_1, z_t \in \Lambda} q(0, z_1) \sum_{\mathcal{T} \in \mathbb{T}(z_1)} \prod_{e \in \mathcal{T}} d(e)^{2-d} \\ &\quad \times \sum_{\mathcal{E} \leftarrow \mathcal{T}} \prod_{e \in \Lambda^2} p(e)^{\mathcal{E}_{\mathcal{S}}(e)} \prod_{z \in \Lambda} \frac{(\sum_{x \in \Lambda} \mathcal{E}_{\mathcal{S}}(z, x))!}{\prod_{x \in \Lambda} \mathcal{E}_{\mathcal{S}}(z, x)!}. \end{aligned} \quad (4.34)$$

Thus, from (4.34), we obtain after summing over  $\mathcal{E}_{\mathcal{S}}$ , and taking into account the degeneracy explained in point 3 above,

$$\begin{aligned}
P_0(l_\infty(z) = n(z), \forall z \in \Lambda) &\leq (ec'_d \bar{n})^{|\Lambda|} |\Lambda| (|\Lambda|!)^{d-1} e^{-\underline{n} \text{cap}(\Lambda)} \sum_{\gamma \in \mathcal{S}(\Lambda)} \prod_{i=1}^{|\Lambda|} d(\gamma(i), \gamma(i-1))^{2-d} \\
&\times \sum_{\mathcal{E} \in \mathbb{N}^{\Lambda \times \Lambda}} \mathbb{I}_{\{n(z) = \sum_x \mathcal{E}(z, x) + 1\{z=z_t\}, \forall z \in \Lambda\}} \prod_{z \in \Lambda} \left( \frac{(\sum_{x \in \Lambda} \mathcal{E}(z, x))!}{\prod_{x \in \Lambda} \mathcal{E}(z, x)!} \prod_{x \in \Lambda} p(z, x)^{\mathcal{E}(z, x)} \right) \\
&\leq (ec'_d \bar{n})^{|\Lambda|} (|\Lambda|!)^d e^{-\underline{n} \text{cap}(\Lambda)} \sum_{\gamma \in \mathcal{S}(\Lambda)} \prod_{i=1}^{|\Lambda|} d(\gamma(i), \gamma(i-1))^{2-d}.
\end{aligned} \tag{4.35}$$

Proposition 1.5 follows now from the (4.35) and the lower bound in (4.32).

## 5 Proof of Corollary 1.7

Assume that  $d \geq 5$ . First, note that

$$\mathbb{P}(|\mathcal{D}(n) \cap \tilde{\mathcal{D}}(m)| \geq L) \leq \sum_{\Lambda \subset \mathbb{Z}^d: |\Lambda|=L} P_0(l_\infty(z) = n, \forall z \in \Lambda) \times P_0(l_\infty(z) = m \forall z \in \Lambda). \tag{5.1}$$

Thus, if we set  $\varphi(x) = \|x\|^{2-d}$  for  $x \in \mathbb{Z}^d$ , Proposition 1.5 yields by Cauchy-Schwarz,

$$\begin{aligned}
\mathbb{P}(|\mathcal{D}(n) \cap \tilde{\mathcal{D}}(m)| \geq L) &\leq \sum_{\Lambda \subset \mathbb{Z}^d: |\Lambda|=L} (c_d n \times m)^{|\Lambda|} e^{-(n+m)\text{cap}(\Lambda)} (|\Lambda|!)^{2d} \\
&\sum_{\gamma \in \mathcal{S}(\Lambda \setminus \{0\})} \prod_{i=1}^{|\Lambda \setminus \{0\}|} \varphi(\gamma(i) - \gamma(i-1))^2,
\end{aligned} \tag{5.2}$$

We first show that there is a constant  $\kappa_d > 0$ , such that for any finite  $\Lambda \subset \mathbb{Z}^d$ , we have

$$\text{cap}(\Lambda) \geq \kappa_d |\Lambda|^{1-\frac{2}{d}}. \tag{5.3}$$

We use the variational characterisation of capacity (see the Appendix of [6]), which says that if  $G_d$  is the Green kernel,  $\mu$  is a non-negative measure on  $\Lambda$ , and  $c$  a positive constant

$$\forall z \in \Lambda, \quad \sum_{z' \in \Lambda} G_d(z, z') \mu(z') \leq c \implies \text{cap}(\Lambda) \geq \frac{\sum_{z \in \Lambda} \mu(z)}{c}. \tag{5.4}$$

We have shown in the proof of Lemma 1.2 of [3], that there is  $\kappa_d > 0$  such that for any  $z \in \Lambda$

$$\sum_{z' \in \Lambda} G_d(z, z') \mu(z') \leq \frac{1}{\kappa_d}, \quad \text{with} \quad \mu(z) = \frac{\mathbb{I}_\Lambda(z)}{|\Lambda|^{2/d}}. \tag{5.5}$$

The desired bound (5.3) follows readily from (5.5).

Now, note that (with the convention  $\gamma(0) = z(0) = 0$ )

$$\sum_{\Lambda: |\Lambda|=L} \sum_{\gamma \in \mathcal{S}(\Lambda)} \prod_{i=1}^{|\Lambda|} \varphi(\gamma(i) - \gamma(i-1))^2 = \sum_{z_1, \dots, z_L \text{ distinct}} \prod_{i=1}^L \varphi(z_i - z_{i-1})^2 \leq \left( \sum_{z \in \mathbb{Z}^d} \varphi(z)^2 \right)^L. \quad (5.6)$$

Now, dimension 5 or more enters in making the last series convergent, and the result follows at once.

## References

- [1] Asselah, A., *Large Deviation Principle for the Self-Intersection Local Times for Simple Random Walk in dimension 5 or more*. Preprint 2007, arXiv:0707.0813.
- [2] Asselah, A., *Large Deviations for the Self-Intersection Times for Simple Random Walk in dimension  $d = 3$* . To appear in Probab. Theory & Related Fields,
- [3] Asselah, A., Castell F., *A note on random walk in random scenery*. Annales de l'I.H.P., 43 (2007) 163-173.
- [4] E. Bolthausen, F. den Hollander and M. van den Berg *On the volume of the intersection of two Wiener sausages* The Annals of Mathematics, Volume 159, Number 2 (2004), 741-782.
- [5] Chen, X. and Morters, P. *Upper tails for intersection local times of random walks in supercritical dimensions*. (preprint 2007)
- [6] Fukai, Y.; Uchiyama, K. *Wiener's test for space-time random walks and its applications* Trans.Amer.Math.soc., vol 348,n 10,(1996), 4131-4152.
- [7] Khanin, K. M.; Mazel, A. E.; Shlosman, S. B.; Sinai, Ya. G. *Loop condensation effects in the behavior of random walks*. The Dynkin Festschrift, 167–184, Progr. Probab., 34, Birkhuser Boston, Boston, MA, 1994.
- [8] Kondo K., Hara T., *Critical exponent of susceptibility for a class of general ferromagnets in  $d > 4$  dimensions* J.Math.Phys.28(5),1987, 1206-1208.
- [9] Lawler, G., *Intersection of Random Walks* Probability and its Applications. Birkhuser Boston, Inc., Boston, MA, 1991.